Int. J. Solids Structures Vol. 25, No. 9, pp. 1077–1100, 1989
Printed in Great Britain.

0020-7683/89 $3.00 + .00

ON THE SPECTRAL THEORY OF LINEAR MECHANICAL STRUCTURES GOVERNED BY FOURTH ORDER SELF-ADJOINT ORDINARY DIFFERENTIAL EQUATIONS

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(*Received* 29 *February* 1988; *in revised form* 15 *February* 1989)

Abstract—Conventionally, normal modes (eigenfunctions) are labeled with regard to the magnitudes of associated eigenfrequencies (eigenvalues). Linear mechanical structures governed by second order ODE's possess a set of the oscillatory properties, two of which are: (i) all eigenfrequencies μ_ks are distinct, and (ii) the kth normal mode has exactly $N_k = k-1$ nodes. Hence, under the conventional rule, μ_ks and N_ks are in full agreement: both sets form strongly increasing sequences.

It is well known that the first oscillatory property fails for linear mechanical structures governed by fourth order ODEs: they may have repeated eigenfrequencies. However, it turns out that the second oscillatory property also fails: several non-consecutive modes may have the same number of nodes. Thus, in this case, the conventional rule may lead to the complete disagreement between both sets: while μ_ks form an increasing (non-descending) sequence, a sequence of N_ks may be disordered. Therefore, (i) higher modes may have a smaller number of nodes than lower modes (in particular, the fundamental mode may have many nodes, while any higher mode may even be nodeless) and, (ii) the normal mode responses, treated as functions of a rigidity (or inertia) parameter of the structure, become discontinuous. The latter disadvantage directly relates to the problem of modal truncation: small changes in mechanical properties of the structure may lead to significant (even complete) changes in the modal responses due to the same excitation.

All such spectral features are studied in the proposed paper with regard to regular continuous beams with elastic interior supports whose relative stiffness is considered as a rigidity parameter.

1. INTRODUCTION

1.1. The general spectral properties of finite-dimensional linear mechanical structures with distributed parameters are: (i) the eigenfrequencies (eigenvalues), μ, are real, positive, and form a denumerable infinite spectrum and (ii) the normal modes (eigenfunctions), Φ, comprise a complete set of orthonormal functions. Let (μ, Φ) be an eigenpair. Conventionally, eigenpairs are labeled with regard to the magnitude of μs: $\Phi = \Phi_k$, if $\mu \equiv \mu_k \geqslant \mu_{k-1} \geqslant \cdots \geq \mu_1$. The essential feature of the normal mode Φ_k is the number of variations of sign in its components, or the number of nodes, N_k, where the nodes are defined by the following conditions

$$\left.\begin{array}{r}\Phi(x) = 0 \\ \Phi(x-0)\Phi(x+0) < 0\end{array}\right\}, \tag{1}$$

and x is a space coordinate. Structures, which are governed by *second* order self-adjoint ordinary differential equations (ODEs) (strings, shafts, or, in general, Sturm–Liouville systems), possess a set of the *oscillatory properties* (Courant and Hilbert, 1953; Gantmacher and Krein, 1950; Gantmacher, 1960), two of which are the base for the following consideration. They are: (i) all eigenfrequencies are distinct, and (ii) the normal mode Φ_k has exactly $k-1$ nodes:†

$$N_k = k-1, \quad k = 1, 2, \ldots \tag{2}$$

For Sturm–Liouville systems the conventional rule for mode numbering accords with the

† As an example of other oscillatory properties, note that the nodes of two adjacent modes alternate (Gantmacher and Krein, 1950; Gantmacher, 1960).

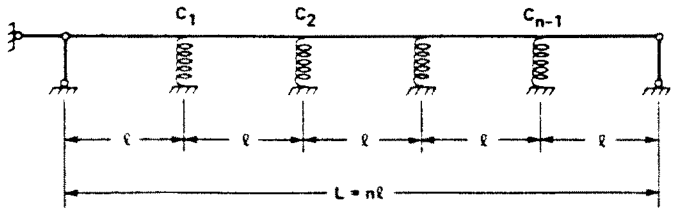


Fig. 1. A regular beam with elastic interior supports (RBES).

numbers of nodes: if we label eigenpairs with regard to the number of nodes, we obtain the same sequence.

1.2. It is well known that the first oscillatory property fails for linear mechanical structures, governed by fourth order ODEs (for example, beam-frame structures): such structures may have multiple eigenfrequencies. However, it turns out that the second oscillatory property (2), also fails: several non-consecutive normal modes may have the same number of nodes. The conventional rule for mode labeling may lead to the complete disagreement between μ_ks and N_ks: while μ_ks form an increasing (non-decreasing) sequence, the sequence of N_ks is not ordered at all. Therefore higher modes may have smaller numbers of nodes than lower modes (in particular, the fundamental mode may have many nodes, while any higher mode may be nodeless) and the modal responses become discontinuous functions of a rigidity (or inertia) parameter of the structure.†

1.3. All these spectral features are the subject of this paper. We shall study them with regard to regular beams with elastic interior supports (RBES), that is, multispan beams with identical spans. We assume that the RBES are linear, have uniformly distributed mass and simply supported extreme ends, Fig. 1. Hence they possess a symmetry group C_2.

We introduce the following notation:

n	is the number of spans,
EI	is the rigidity of the beam, EI = const.,
l	is the span length, l = const.,
$L = nl$	is the total beam length,
ψ	is the stiffness of the elastic support (force per length),
$\bar{\psi}$	is the relative (dimensionless) stiffness of the elastic support, $\bar{\psi} \in (0, \infty)$,

$$\bar{\psi} = \psi l^3/EI, \tag{3}$$

m	is the beam mass per unit length,
ω_k	is the kth natural frequency (rad/s),
μ_k	is the kth eigenfrequency,

$$\mu_k = (m\omega_k^2 l^4/EI)^{1/4} \tag{4}$$

Φ_k	is the kth normal mode, normalized to max, $\Phi_k(x) = 1$, $0 \leq x \leq L$.

Forced vibration of the RBES are described by:

(a) n fourth order PDEs

$$EI\frac{\partial^4 u(x,t)}{\partial x^4} + b\frac{\partial u(x,t)}{\partial t} + m\frac{\partial^2 u(x,t)}{\partial t^2} = p(x,t), \quad x \in [0, l] \tag{5}$$

where b is the viscous damping and $p(x, t)$ is the excitation force,

(b) the boundary conditions

† Two-dimensional structures (for instance, membranes) may have repeated eigenfrequencies even when they are described by second order ODEs. However it is apparent that they are not included in a class of structures under investigation because their normal modes have no nodal points but nodal lines. On the other hand, 2D and 3D lattices belong to the structures under consideration.

$$u(0,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = 0, \quad u(L,t) = \frac{\partial^2 u(L,t)}{\partial x^2} = 0, \tag{6}$$

(c) $n-1$ compatability conditions (last two are moment and shear balance, respectively) at the elastic support points $C_j, j = 1, \ldots, n-1$

$$u|_{C_j-0} = u|_{C_j+0}, \qquad \left.\frac{\partial u}{\partial x}\right|_{C_j-0} = \left.\frac{\partial u}{\partial x}\right|_{C_j+0}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{C_j-0} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{C_j+0}, \quad \left.\frac{\partial^3 u}{\partial x^3}\right|_{C_j-0} = \left.\frac{\partial^3 u}{\partial x^3}\right|_{C_j+0} + \frac{\psi}{EI} u|_{C_j+0} \tag{7}$$

and (d) the initial conditions

$$u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0. \tag{8}$$

Let the RBES be subjected to the excitation force of the form

$$p(x,t) = q(x) f(t), \tag{9}$$

where $q(x)$ is the force distribution and $f(t)$ is the forcing time-function. One can present the RBES displacements in the form given in Meirovitch (1967) and Biggs (1964):

$$u(x,t) = \sum_{k=1}^{r} \frac{\Gamma_k}{\omega_k^2} \Phi_k(x) F_k(t), \quad x \in [0, L]. \tag{10}$$

Here $F_k(t)$ is the kth dynamic load factor

$$F_k(t) = \frac{\omega_k}{\sqrt{1-\beta^2}} \int_0^t f(\tau)\, e^{-\beta(t-\tau)} \sin\left(\omega_k \sqrt{1-\beta^2}(t-\tau)\right) d\tau, \tag{11}$$

$\beta = b/2m$ is the critical damping, Γ_k is the participation factor of the kth normal mode $\Phi_k(x)$

$$\Gamma_k = Q_k/M_k, \tag{12}$$

Q_k is the kth modal force

$$Q_k = \int_0^L q(x)\Phi_k(x)\, dx, \tag{13}$$

and M_k is the kth modal mass

$$M_k = m\int_0^L \Phi_k^2(x)\, dx. \tag{14}$$

It is necessary to emphasize that the normal modes, Φ_k, are undamped modes (Meirovitch, 1967), since the boundary-value problem, which we obtain by separating space and time variables, describes the undamped vibration, while the damping term is related to the initial-value problem and is revealed in (11).

1.4. Apparently, the degree to which normal modes participate in the total response depends on the force distribution $q(x)$, the dynamic load factor $F_k(t)$, and the spectral properties of the system. Simplifying the problem to the invariant part of the response, we assume that

$$q(x) = q = \text{const.} \quad \text{and} \quad F = \max_{k,t} F_k(t) \tag{15}$$

(skew-symmetric forces can be treated in a similar fashion). Then

$$\Gamma_k = \frac{q}{m}\Gamma_{0,k}, \quad k = 1, 2, \ldots \tag{16}$$

where $\Gamma_{0,k}$ is a dimensionless participation factor, which we call the unweighted participation factor

$$\Gamma_{0,k} = \int_0^L \Phi_k(x)\,\mathrm{d}x \Big/ \int_0^L \Phi_k^2(x)\,\mathrm{d}x, \quad k = 1, 2, \ldots \tag{17}$$

Thus

$$u(x,t) = \left[\sum_{k=1}^{\infty} \Gamma_{4,k}\Phi_k(x)\right] F\frac{ql^4}{EI}, \tag{18}$$

where $\Gamma_{4,k}$ is the weighted participation factor

$$\Gamma_{4,k} = \Gamma_{0,k}/\mu_k^4, \quad k = 1, 2, \ldots \tag{19}$$

We define the amplitudes, $w(x)$, as root-mean-square values

$$w(x) \leqslant \left[\sum_{k=1}^{\infty} (\Gamma_{4,k}\Phi_k(x))^2\right]^{1/2} F\frac{ql^4}{EI}, \tag{20}$$

then

$$w_{\max} = \Gamma_4 F\frac{ql^4}{EI}, \tag{21}$$

where

$$\Gamma_4 = \left[\sum_{k=1}^{\infty} \Gamma_{4,k}^2\right]^{1/2}. \tag{22}$$

In the same manner the amplitudes of bending moments and shears are

$$M(x) = -\left[\sum_{k=1}^{\infty} (\Gamma_{2,k}\Phi_k^{(2)}(x))^2\right]^{1/2} Fql^2,$$

$$V(x) = -\left[\sum_{k=1}^{\infty} (\Gamma_{1,k}\Phi_k^{(3)}(x))^2\right]^{1/2} Fql, \tag{23}$$

where

$$\Phi_k^{(2)}(x) = \frac{l^2}{\mu_k^2}\Phi_k''(x), \quad \Phi_k^{(3)}(x) = \frac{l^3}{\mu_k^3}\Phi_k'''(x), \quad k = 1, 2, \ldots$$

and

$$\Gamma_{2,k} = \Gamma_{0,k}/\mu_k^2, \quad \Gamma_{1,k} = \Gamma_{0,k}/\mu_k, \quad k = 1, 2, \ldots \tag{24}$$

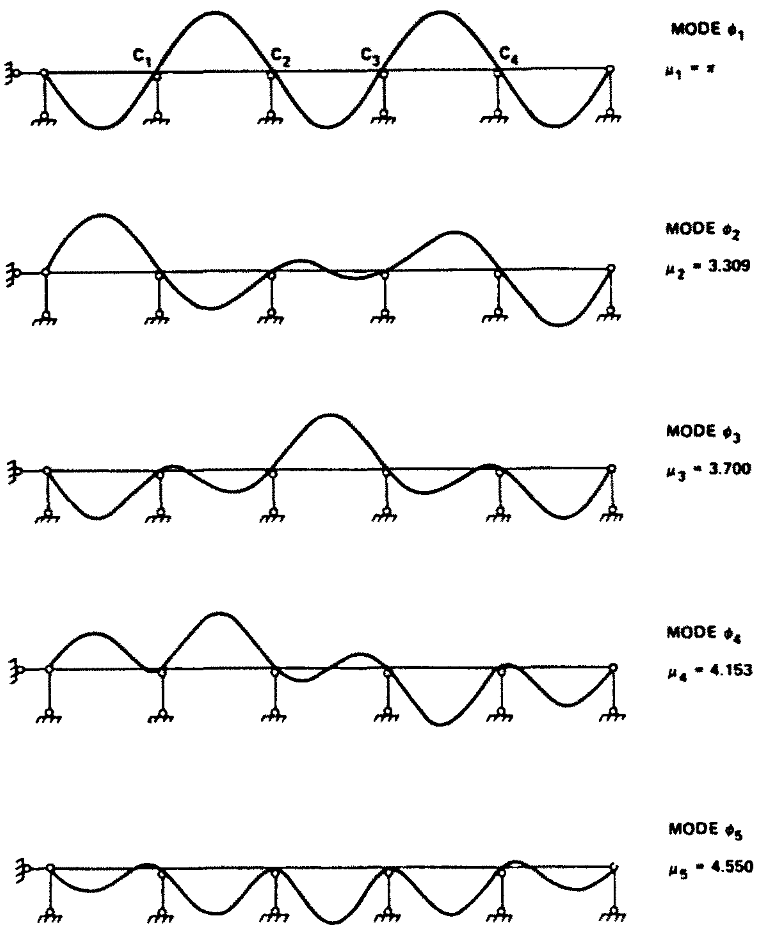


Fig. 2. Normal modes of a five-span RBES.

Clearly

$$M_{max} = \Gamma_2 Fql^2, \quad V_{max} = \Gamma_1 Fql \tag{25}$$

and

$$\Gamma_2 = \left[\sum_{k=1}^{\infty} \Gamma_{2,k}^2\right]^{1/2}, \quad \Gamma_1 = \left[\sum_{k=1}^{\infty} \Gamma_{1,k}^2\right]^{1/2}. \tag{26}$$

Thus, there are four participation factors, $\Gamma_{p,k}$, $p = 0, 1, 2$ and 4, and three total response functions Γ_p, $p = 1, 2, 4$. (One can add Γ_3 and $\Gamma_{3,k} = \Gamma_{0,k}/\mu_k^3$, associated with the angular displacements.) We define the responses of the normal mode Φ_k by

$$\gamma_{p,k} = [\Gamma_{p,k}/\Gamma_p]^2, \quad p = 0, 1, 2 \text{ and } 4; \quad k = 1, 2, \ldots \tag{27}$$

They are the *invariant parts* of actual modal responses. As well as μ_k and Φ_k, $\gamma_{p,k}$ are functions of $\bar{\psi}$ and n.

1.5. One-span beams possess a complete set of oscillatory properties. The spectral properties of arbitrary multispan beams with rigid interior supports were studied by Gantmacher and Krein (1950) and Gantmacher (1960), who have developed a strong mathematical technique; namely, the theory of oscillatory matrices (for lumped-mass systems) and the theory of oscillatory kernels of integral equations (for distributed-mass systems). They have found that arbitrary multispan beams with rigid supports also possess the oscillatory properties under the special modification of the rule for node counting. To illustrate this modification, consider five lowest normal modes of a five-span regular beam with rigid interior supports (RBRS), Fig. 2. Clearly, if we define nodes by (1), then $N_1 = 4$,

$N_2 = 5$, $N_3 = 6$, $N_4 = 7$ and $N_5 = 4$. Gantmacher and Krein proposed to ignore the nodes at those support points C_j, where the normal modes change sign, and to count as nodes those points C_j, where modes have zero slope. According to this rule, $N_1 = 0$, $N_2 = 1$, $N_3 = 2$, $N_4 = 3$ and $N_5 = 4$. In the latter case we ignore points C_1 and C_4 but count C_2 and C_3. (Note that in fact at the area of point C_2 (and point C_3) there are two node points located very near each other.)

With respect to $\bar{\psi}$ the RBES are located between a one-span beam ($\bar{\psi} = 0$) and the RBRS ($\bar{\psi} = \infty$). Since both extreme cases possess the oscillatory properties, it would be natural to expect that the RBES also have them. However, this is not true, in particular, because the modified rule cannot be applied to the RBES: if $\bar{\psi} < 10^4$, displacements of the elastic support points C_j are not ignorable.

1.6. We shall study the spectral properties of the RBES and the RBES modal response functions. In the next section we construct decoupled frequency equations of the RBES and study the distribution of their roots. These equations allow us to introduce a special non-conventional rule for eigenpairs labeling. In Section 3 we investigate the RBES spectral properties using both conventional and non-conventional sequences of eigenpairs. Modal response functions are analyzed in Section 4 and a final discussion is presented in Section 5.

2. FREQUENCY EQUATION OF THE RBES

2.1. To solve the boundary-value problem for the RBES one would usually use one of two classical methods of structural dynamics: the dynamic flexibility method or the dynamic stiffness method. However, being applied to the RBES, they lead to systems of equations which cannot be diagonalized or block diagonalized explicitly. Therefore, we use the mixed method and apply two mixed unknown amplitudes, namely, the bending moment M_j, and the displacement $-w_j$, at each elastic support point C_j, $j = 1, \ldots, n-1$. (Mixed pairs (M_j, w_j) lead to nonsymmetric systems of equations.) The vector of unknowns X_*, is of order $2(n-1)$ and contains $n-1$ second order subvectors

$$X_*^{\mathrm{T}} = [X_1^{\mathrm{T}}, \ldots, X_{n-1}^{\mathrm{T}}], \quad X_j^{\mathrm{T}} = [M_j, -w_j]$$

Subscript "*" stands for block vectors and block matrices. The corresponding system of linear equations is homogeneous and symmetric

$$A_*(\mu)X_* = \mathbb{O}$$

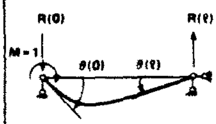
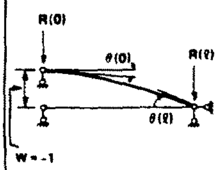
The matrix $A_*(\mu)$ is symmetric tri-block diagonal finite Toeplitz matrix (i.e. $A_{ij} = A_k$, where $k = |i-j|$; $i, j = 1, \ldots, n-1$):

$$A_*(\mu) = \begin{bmatrix} A_0(\mu) & A_1(\mu) & & & \\ A_1(\mu) & A_0(\mu) & A_1(\mu) & & \\ & A_1(\mu) & A_0(\mu) & A_1(\mu) & \\ & & \cdots\cdots & & \\ & & & A_1(\mu) & A_0(\mu) \end{bmatrix} \tag{28}$$

with blocks

$$A_0(\mu) = 2\left[\begin{array}{c|c} \dfrac{l}{EI}\alpha_\mu & \dfrac{1}{l}\delta_\mu \\ \hline \dfrac{1}{l}\delta_\mu & \dfrac{EI}{l^3}(\gamma_\mu - \tfrac{1}{2}\bar{\psi}) \end{array}\right] \tag{29}$$

Table 1. Beam functions α_μ, β_μ, δ_μ, ε_μ, γ_μ and η_μ

#	SCHEME	FORMULA	FUNCTION
1		$\theta(0) = \frac{\ell}{EI}\alpha_\mu$	$\alpha_\mu = \frac{1}{\mu}\frac{\sin\mu\cosh\mu - \cos\mu\sinh\mu}{2\sin\mu\sinh\mu}$
		$\theta(\ell) = \frac{\ell}{EI}\beta_\mu$	$\beta_\mu = \frac{1}{\mu}\frac{\sinh\mu - \sin\mu}{2\sin\mu\sinh\mu}$
		$R(0) = \frac{1}{\ell}\delta_\mu$	$\delta_\mu = \mu\frac{\sin\mu\cosh\mu + \cos\mu\sinh\mu}{2\sin\mu\sinh\mu}$
		$R(\ell) = \frac{1}{\ell}\varepsilon_\mu$	
2		$\theta(0) = \frac{1}{\ell}\delta_\mu$	$\varepsilon_\mu = \mu\frac{\sinh\mu + \sin\mu}{2\sin\mu\sinh\mu}$
		$\theta(\ell) = \frac{1}{\ell}\varepsilon_\mu$	
		$R(0) = \frac{EI}{\ell^3}\gamma_\mu$	$\gamma_\mu = \mu^3\frac{\sin\mu\cosh\mu - \cos\mu\sinh\mu}{2\sin\mu\sinh\mu}$
		$R(\ell) = \frac{EI}{\ell^3}\eta_\mu$	$\eta_\mu = \mu^3\frac{\sinh\mu - \sin\mu}{2\sin\mu\sinh\mu}$

and

$$A_1(\mu) = \left[\begin{array}{c|c} \frac{l}{EI}\beta_\mu & -\frac{1}{l}\varepsilon_\mu \\ \hline -\frac{1}{l}\varepsilon_\mu & \frac{EI}{l^3}\eta_\mu \end{array}\right]. \tag{30}$$

Elements of the first row of $A_0(\mu)$ are the angular displacements, θ_j, at points C_j, $j = 1, \ldots, n-1$, induced by $M_j = 1$ and $w_j = -1$, respectively. Elements of its second row are the reaction forces R_j, at C_j, induced by the same sources. Elements of blocks $A_1(\mu)$ are the angular displacements and the reaction forces at points C_{j-1} and C_{j+1}, induced by $M_j = 1$ and $w_j = -1$. Functions α_μ, β_μ, δ_μ, ε_μ, γ_μ and η_μ are meromorphic hyperbola-trigonometric functions of μ given in Table 1.

2.2. In accordance with Dinkevich (1986), matrix $A_*(\mu)$ has the following explicit block diagonal decomposition

$$A_*(\mu) = U_*\Lambda_*(\mu)U_*. \tag{31}$$

Here $\Lambda_*(\mu)$ is block diagonal

$$\Lambda_*(\mu) = \begin{bmatrix} \Lambda_1(\mu) & & & \\ & \Lambda_2(\mu) & & \\ & & \cdots\cdots & \\ & & & \Lambda_{n-1}(\mu) \end{bmatrix} \tag{32}$$

with 2×2 blocks

$$\Lambda_i(\mu) = A_0(\mu) + 2A_1(\mu)\cos\theta_i = 2\left[\begin{array}{c|c} \frac{l}{EI}(\alpha_\mu + \beta_\mu\cos\theta_i) & \frac{1}{l}(\delta_\mu - \varepsilon_\mu\cos\theta_i) \\ \hline \frac{1}{l}(\delta_\mu - \varepsilon_\mu\cos\theta_i) & \frac{EI}{l^3}(\gamma_\mu + \eta_\mu\cos\theta_i - \frac{1}{2}\bar{\psi}) \end{array}\right] \tag{33}$$

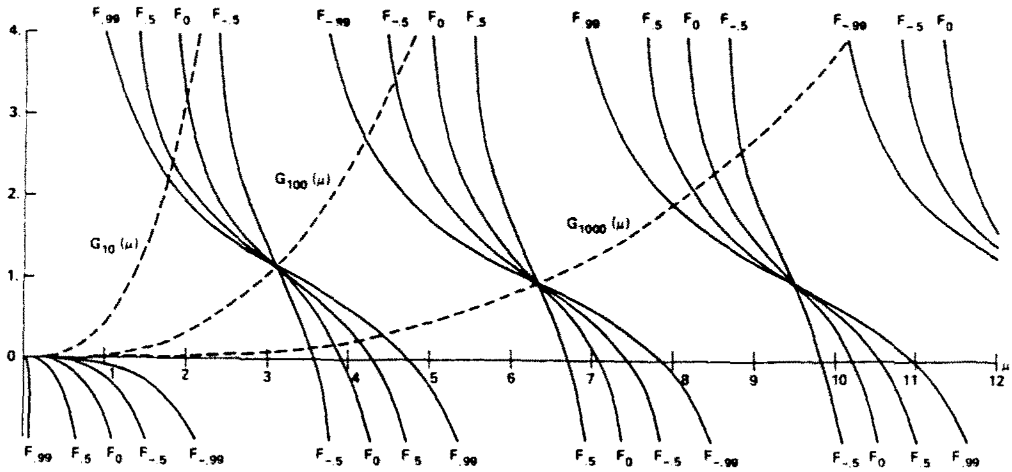


Fig. 3. Functions $F_{\cos\theta_i}(\mu)$ and $G_{\bar{\psi}}(\mu)$.

$$\theta_i = i\pi/n, \quad i = 1, 2, \ldots, n-1. \tag{34}$$

Matrix U_* is symmetric, orthonormal and independent of μ

$$U_* = \sqrt{\frac{2}{n}}[I_2 \sin ij\pi/n]_{i,j=1}^{n-1}, \quad U_*^{-1} = U_*. \tag{35}$$

I_2 is a 2×2 unit matrix. Thus, the RBES frequency equation

$$\det A_*(\mu) = 0 \tag{36}$$

reduces to $n-1$ uncoupled equations.

$$\det \Lambda_i(\mu) = (\alpha_\mu + \beta_\mu \cos\theta_i)(\gamma_\mu + \eta_\mu \cos\theta_i - \tfrac{1}{2}\bar{\psi}) - (\delta_\mu - \varepsilon_\mu \cos\theta_i)^2 = 0,$$
$$i = 1, 2, \ldots, n-1. \tag{37}$$

Introducing here the expressions for $\alpha_\mu, \ldots, \eta_\mu$ (Table 1), one can write the frequency equations in the form

$$\frac{\sinh\mu}{\cosh\mu - \cos\theta_i} - \frac{\sin\mu}{\cos\mu - \cos\theta_i} = \frac{4\mu^3}{\bar{\psi}}, \quad \theta_i = i\pi/n, \quad i = 1, 2, \ldots, n-1 \tag{38}$$

These equations were obtained by Leites (1974), who applied a finite-difference technique for their derivation.

2.3. Denote by $F_{\cos\theta_i}(\mu)$ and $G_{\bar{\psi}}(\mu)$ the left and the right sides of (38), respectively. These functions are plotted in Fig. 3. Similar to $\tan\mu$, each function $F_{\cos\theta_i}(\mu)$ has a denumerable infinite number of segments divided by poles. The function $G_{\bar{\psi}}(\mu)$ is a cubic parabola of μ scaled down by a factor of $1/\bar{\psi}$. The points, where $G_{\bar{\psi}}(\mu)$ intersects $F_{\cos\theta_i}(\mu)$, determine the roots of the ith frequency equation (38): $\mu_1^{(i)}(\bar{\psi}) < \mu_2^{(i)}(\bar{\psi}) < \ldots$. We observe from Fig. 3 that parabola $G_{\bar{\psi}}(\mu)$ crosses the first segment of all curves $F_{\cos\theta_i}(\mu)$, $i = 1, \ldots, n-1$, before it will meet the second segment of any $F_{\cos\theta_i}(\mu)$. This permits one to group all roots of all frequency eqns (38) into a denumerable infinite number of sets so that the pth set would contain only the pth root of each eqn (38): $\mu_p^{(1)}(\bar{\psi}), \mu_p^{(2)}(\bar{\psi}), \ldots, \mu_p^{(n-1)}(\bar{\psi})$, $p = 1, 2, \ldots$. They are known as *zones of the natural frequency condensation*.

2.4. However, these zones are incomplete because the roots of (38) do not span the entire spectrum of the RBES. In fact, the original frequency equation (36) determines only those eigenfrequencies which correspond to nontrivial vectors X_*. We call them the *explicit* eigenfrequencies, Dinkevich (1977). The RBES also have such normal modes where $M_j = w_j = 0$ at all elastic support points C_j, $j = 1,2,\ldots,n-1$. They correspond to trivial vectors $X_* = \mathbb{O}$. According to these modes each span vibrates as an independent simply supported beam with one, two, etc., half-waves, Fig. 2, mode Φ_1. The associated eigenfrequencies are $\pi, 2\pi, \ldots$. We call them the *implicit* eigenfrequencies, since eqns (38) do not "see" them. It should be natural to insert $\mu = \pi$ into the first zone of condensation, $\mu = 2\pi$ into the second zone and so on. Thus, the complete zones contain n eigenfrequencies each, and this is true for any multispan beam. In our case one eigenfrequency is implicit, while all others of the same zone are explicit.

2.5. Further examination of Fig. 3 reveals the existence of the points at which curves $F_{\cos\theta_i}(\mu)$, $i = 1,\ldots,n-1$, intersect each other. Since $G_{\bar\psi}(\mu)$ may run through any of those points, there exists such values of $\bar\psi$, which imply multiple eigenfrequencies.

2.6. Setting $\bar\psi \to \infty$, we obtain the frequency equations of the RBRS

$$\frac{\sinh \mu}{\cosh \mu - \cos \theta_i} - \frac{\sin \mu}{\cos \mu - \cos \theta_i} = 0, \quad \theta_i = i\pi/n, \quad i = 1,\ldots,n. \tag{39}$$

The roots of (39) are the points in which curves $F_{\cos\theta_i}(\mu)$ intersect the horizontal axis μ. Clearly, they are distinct. Spectral properties of the RBRS were studied in Dinkevich (1974).

2.7. Having obtained $\mu^{(i)}$, $i = 1,2,\ldots$, we find the associated normal modes. In accordance with (31)–(35), the modal values of $M_j^{(i)}$ and $w_j^{(i)}$ at each elastic support point C_j, $j = 1,\ldots,n-1$, may be presented in the following form:

$$M_j^{(i)}(\bar\psi) = \frac{1}{l}\sqrt{\frac{2}{n}}\left[\delta(\mu^{(i)}) - \varepsilon(\mu^{(i)}) \cos\frac{i\pi}{n}\right] \sin\frac{ij\pi}{n}$$

$$w_j^{(i)}(\bar\psi) = \frac{EI}{l^3}\sqrt{\frac{2}{n}}\left[\gamma(\mu^{(i)}) + \eta(\mu^{(i)}) \cos\frac{i\pi}{n} - \tfrac{1}{2}\bar\psi\right] \sin\frac{ij\pi}{n}$$

$$i = 1,2,\ldots; \quad j = 1,\ldots,n-1. \tag{40}$$

Using them as boundary conditions, one can find the modal displacement

$$w_j^{(i)}(\xi) = C_1 \sin \mu^{(i)}\xi + C_2 \cos \mu^{(i)}\xi + C_3 \sinh \mu^{(i)}\xi + C_4 \cosh \mu^{(i)}\xi \tag{41}$$

at any specified point $\xi \in [0, 1]$ along the jth span, $j = 1,\ldots,n$. The total set of $w_j^{(i)}(\xi)$ forms the explicit normal mode $\Phi^{(i)}$, $i < n$, which we normalize to $\max_{0 \leqslant x \leqslant L} \Phi^{(i)}(x) = 1$. Clearly, the sequence of eigenpairs $(\mu^{(i)}, \Phi^{(i)})$, $i = 1,2,\ldots$, is unconventional since $\mu^{(i)}$ may be larger than $\mu^{(i+1)}$. It will be called the *physical* (or natural) sequence of eigenpairs.

We will study the properties of both sequences, physical and *conventional* (μ_k, Φ_k), $k = 1,2,\ldots$, and with no loss in generality this consideration is limited to the first zone of frequency condensation ($1 \leqslant i,k \leqslant n$).

3. SPECTRAL FEATURES OF THE RBES

3.1. For any fixed n, explicit $\mu^{(i)}(\bar\psi)$, $i < n$, are monotone continuous functions growing from the eigenfrequencies of a simply supported beam, $\mu^{(i)}(0) = i\pi/n$ (not $i\pi$, since μ (4) is

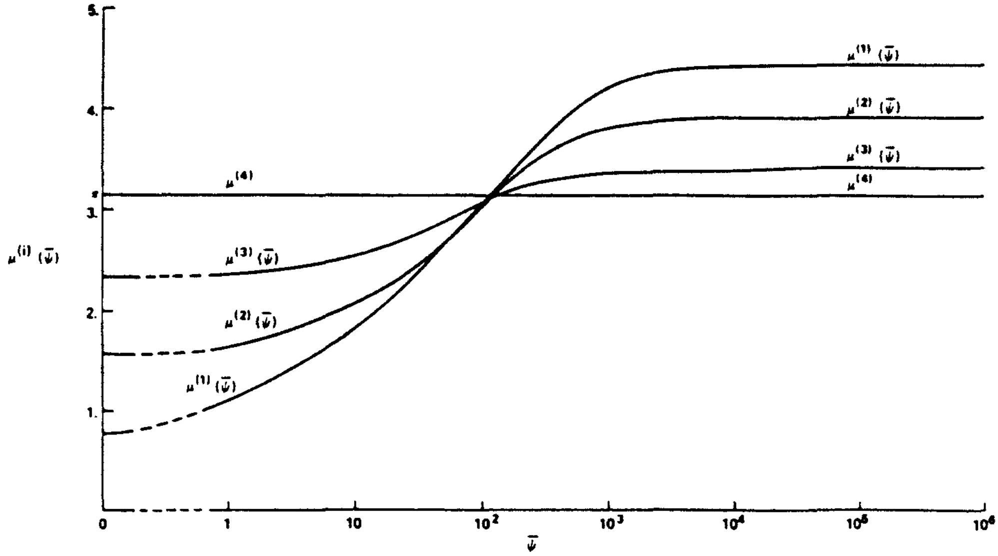


Fig. 4. Functions $\mu^{(i)}(\bar{\psi})$, $n = 4$.

written with regard to a span length $l = L/n$), to their asymptotic values $\mu^{(i)}(\infty)$, associated with the RBRS. They are shown in Fig. 4 for $n = 4$.

Theorem 1. There are $n_* = n(n-1)/2$ values of $\bar{\psi}$, namely, $\bar{\psi}_{ii_1}$, $i_1 > i = 1, \ldots, n-1$, such that

$$\left.\begin{aligned} \mu^{(i)}(\bar{\psi}_{ii_1}-0) &< \mu^{(i_1)}(\bar{\psi}_{ii_1}-0) \\ \mu^{(i)}(\bar{\psi}_{ii_1}) &= \mu^{(i_1)}(\bar{\psi}_{ii_1}) \\ \mu^{(i)}(\bar{\psi}_{ii_1}+0) &> \mu^{(i_1)}(\bar{\psi}_{ii_1}+0) \end{aligned}\right\}. \tag{42}$$

They are determined by the formula

$$\bar{\psi}_{ii_1} = 4\mu_{ii_1}^3 \frac{(\cos\mu_{ii_1}-\cos\theta_i)(\cosh\mu_{ii_1}-\cos\theta_i)}{\sinh\mu_{ii_1}(\cos\mu_{ii_1}-\cos\theta_i)-\sin\mu_{ii_1}(\cosh\mu_{ii_1}-\cos\theta_i)}, \tag{43}$$

where μ_{ii_1} is the first root of the equation

$$\frac{\sinh\mu}{(\cosh\mu-\cos\theta_i)(\cosh\mu-\cos\theta_{i_1})} = \frac{\sin\mu}{(\cos\mu-\cos\theta_i)(\cos\mu-\cos\theta_{i_1})},$$

$$\theta_i = i\pi/n, \quad i_1 > i = 1, \ldots, n-1 \tag{44}$$

Proof. Equation (44) follows from (38), if we subtract one eqn (38) from another; eqn (43) is the other form of (38) with μ_{ii_1} substituted for μ. □

We will call $\bar{\psi}_{ii_1}$ the *double frequency points*. As follows from Table 2, in higher zones double frequency points become very close to each other permitting us to treat them as one m-fold frequency point. Introduce

$$\bar{\psi}_1^{cr} = \bar{\psi}_{12} \quad \text{and} \quad \bar{\psi}_2^{cr} = \bar{\psi}_{n-1,n} \tag{45}$$

and subdivide the entire $\bar{\psi}$-domain $[0, \infty)$, into three subdomains

Table 2. Double Frequency Points $\bar{\psi}_{ii_1}$, $n = 4$

ZONE OF CONDENSATION	i_1	$\bar{\psi}_{ii_1}$ ($n = 4$) $1i_1$	$2i_1$	$3i_1$
1	2	87.29		
	3	108.96	118.76	
	4	116.90	124.49	132.08
2	2	993.81		
	3	996.80	1001.53	
	4	998.04	1003.12	1008.16
3	2	3347.5		
	3	3348.1	3348.5	
	4	3348.3	3348.7	3349.1

$$[0, \bar{\psi}_1^{cr}), [\bar{\psi}_1^{cr}, \bar{\psi}_2^{cr}], (\bar{\psi}_2^{cr}, \infty). \tag{46}$$

If $\bar{\psi}$ belongs to subdomain 1, $\mu^{(i)}(\bar{\psi})$ are distinct and form an increasing sequence

$$\mu^{(1)}(\bar{\psi}) < \mu^{(2)}(\bar{\psi}) < \cdots < \mu^{(n)} = \pi. \tag{47}$$

When $\bar{\psi}$ exceeds $\bar{\psi}_2^{cr}$ (subdomain 3), $\mu^{(i)}(\bar{\psi})$ are also distinct but form a decreasing sequence

$$\mu^{(1)}(\bar{\psi}) > \mu^{(2)}(\bar{\psi}) > \cdots > \mu^{(n)} = \pi. \tag{48}$$

In subdomain 2 any two eigenfrequencies $\mu^{(i)}(\bar{\psi})$, $i = 1, \ldots, n$, may be equal and their sequence is not ordered, Fig. 5a.

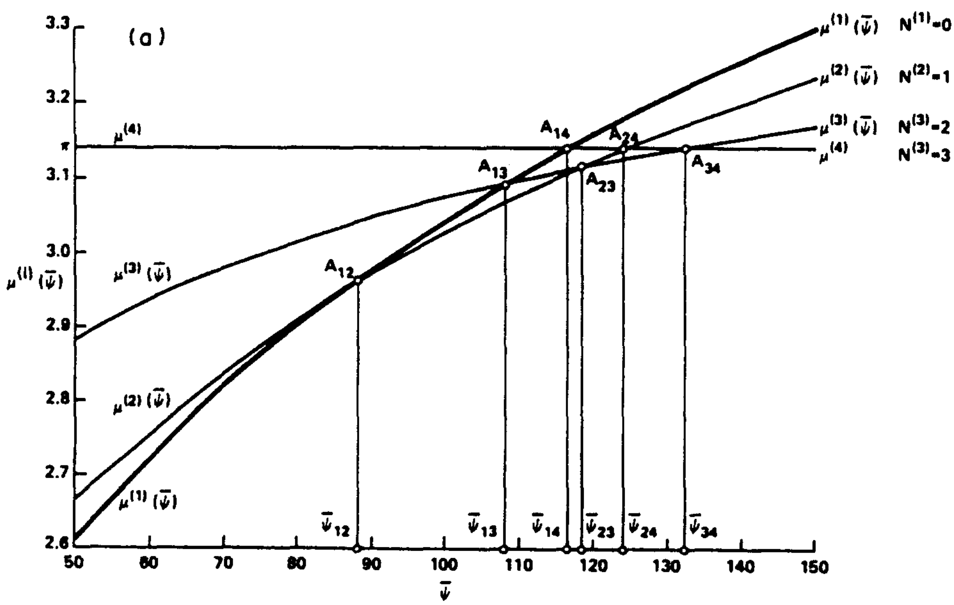


Fig. 5a. Functions $\mu^{(i)}(\bar{\psi})$, $n = 4$, subdomains 1 and 2.

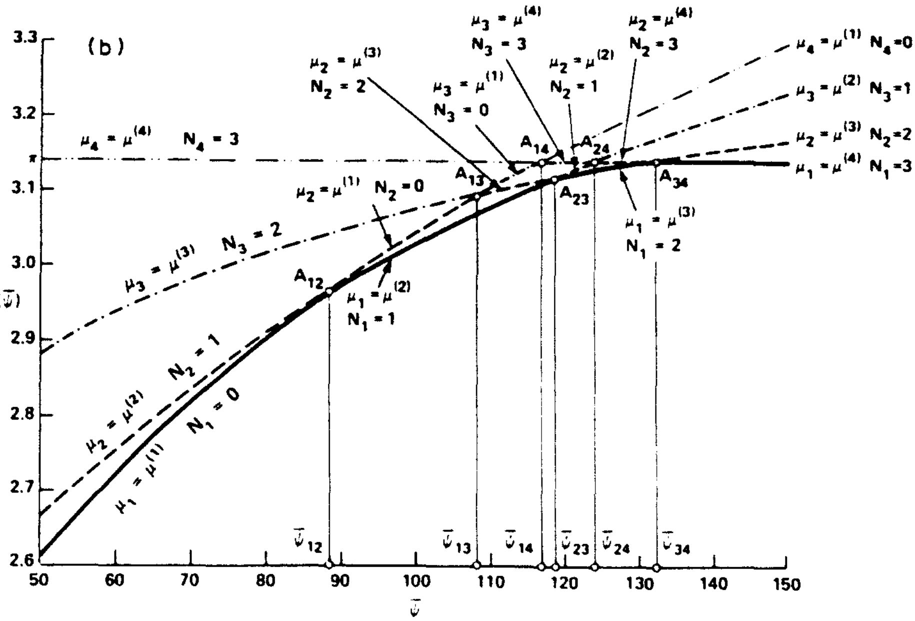


Fig. 5b. Functions $\mu_k(\bar\psi)$, $n = 4$, subdomains 1 and 2.

Apparently $\bar\psi_1^{cr}$ and $\bar\psi_2^{cr}$ vary with n:

$$\bar\psi_1^{cr}(n) = 4\mu_{12}^3 \frac{\left(\cos\mu_{12}-\cos\frac{\pi}{n}\right)\left(\cosh\mu_{12}-\cos\frac{\pi}{n}\right)}{\sinh\mu_{12}\left(\cos\mu_{12}-\cos\frac{\pi}{n}\right)-\sin\mu_{12}\left(\cosh\mu_{12}-\cos\frac{\pi}{n}\right)}, \tag{49}$$

where μ_{12} is the first root of

$$\frac{\sinh\mu}{\left(\cosh\mu-\cos\frac{\pi}{n}\right)\left(\cosh\mu-\cos\frac{2\pi}{n}\right)} = \frac{\sin\mu}{\left(\cos\mu-\cos\frac{\pi}{n}\right)\left(\cos\mu-\cos\frac{2\pi}{n}\right)}.$$

Clearly, $\mu_{n-1,n} = \mu^{(n)} = \pi$, hence (Leites, 1974),

$$\bar\psi_2^{cr}(n) = 4\pi^3\left(\cosh\pi+\cos\frac{\pi}{n}\right)\Big/\sinh\pi. \tag{50}$$

Thus, $\bar\psi_1^{cr}(\infty) = 0$, $\bar\psi_2^{cr}(\infty) = 135.2$, Table 3.

3.2. As mentioned above, the RBES possess a symmetry group C_2, hence their normal modes are either symmetric or skew-symmetric. At $\bar\psi = 0$ the explicit mode $\Phi^{(i)}(0)$, $i < n$,

Table 3. Critical double frequency points $\bar\psi_1^{cr}(n)$ and $\bar\psi_2^{cr}(n)$

n	2	3	4	5	6	7	8	9	10	15	20	∞
$\bar\psi_1^{cr}$	124.5	106.5	87.3	72.7	61.9	53.7	47.4	42.3	38.2	25.7	19.3	0
$\bar\psi_2^{cr}$	124.5	129.9	132.1	133.2	133.8	134.2	134.4	134.6	134.7	135.0	135.1	135.2

is sinusoidal with $i-1$ half-waves. When $\bar{\psi}$ runs through the entire domain $[0, \infty)$, each explicit mode $\Phi^{(i)}(\bar{\psi})$ is deformed (Fig. 6) due to the restoring reaction forces

$$R_j^{(i)}(\bar{\psi}) = -\psi w_j^{(i)}(\bar{\psi}) = -\bar{\psi}\left[\frac{EI}{l^3} w_j^{(i)}(\bar{\psi})\right], \quad i, j = 1, \ldots, n-1 \tag{51}$$

which are applied at each elastic support point C_j and act in the direction of the original straight axis of the beam. Being directly proportional to modal displacements (at points C_j), $R_j^{(i)}(\bar{\psi})$ are either symmetric ($i = 1, 3, 5, \ldots$) or skew-symmetric ($i = 2, 4, 6, \ldots$). This leads to:

Theorem 2. (i) The explicit normal modes $\Phi^{(i)}(\bar{\psi})$, $i < n$, preserve their initial symmetry (associated with $\bar{\psi} = 0$) in the entire domain.

(ii) The implicit mode $\Phi^{(n)}$ is sinusoidal with $n-1$ half-waves, the eigenpair $(\mu^{(n)} = \pi, \Phi^{(n)})$ does not depend on $\bar{\psi}$.

Until $\bar{\psi} \leqslant \bar{\psi}_2^{cr}$ all $w_j^{(i)}(\bar{\psi})$ remain large, hence the initial number of nodes, $N^{(i)}(0)$, also remains unchanged. When $\bar{\psi} > \bar{\psi}_2^{cr}$, in fact $\bar{\psi} \gg \bar{\psi}_2^{cr}$, the magnitudes of $w_j^{(i)}(\bar{\psi})$ become small, and additional nodes may appear with further increase of $\bar{\psi}$.

There are two kinds of mechanisms forming new nodes. The first one is general and may be revealed in any span except the middle. Symmetric and skew-symmetric modes may display this mechanism repeatedly. Suppose that at the neighborhood of point C_j there is a point, say, $C_j \pm \Delta x$, whose amplitude is smaller than $w_j^{(i)}(\bar{\psi})$. Then zero displacement will be achieved first at this point so that any additional increase in $\bar{\psi}$ will replace the "node" (in fact, the point with zero slope) at $C_j \pm \Delta x$ with two new nodes on each side in the neighborhood of this point. One of them will eventually move to C_j as $\bar{\psi}$ continues to grow. Because of the RBES symmetry, this mechanism works simultaneously in two conjugate (symmetrically located) spans. Moreover, it may take place simultaneously in several pairs of conjugate spans, hence

$$N^{(i)}(\bar{\psi}+\Delta\bar{\psi}) = N^{(i)}(\bar{\psi})+4p, \quad p = 1, 2, 3, \ldots \tag{52}$$

For example, for $n = 12$, $N^{(3)}(2410) = 2$ but $N^{(3)}(2411) = 14$, i.e. $p = 3$ and $N^{(4)}(4225) = 3$ but $N^{(4)}(4226) = 19$, i.e. $p = 4$. Another mechanism acts (and only once) in the middle span (n is odd) of skew-symmetric modes $\Phi^{(i)}(\bar{\psi})$, $i = 2, 4, \ldots$. In this case the restoring forces at both ends of the middle span have the same magnitude but opposite signs. Therefore, as $\bar{\psi}$ increases, two new nodes appear simultaneously in this span on the opposite sides of the initial node point at $x = L/2$, see Figs 6b and 6d, thus,

$$N^{(i)}(\bar{\psi}+\Delta\bar{\psi}) = N^{(i)}(\bar{\psi})+2. \tag{53}$$

This mechanism, if it occurs, always preceeds the first one.

Summarizing this observation, we note (see Table 4) that (i) if n is odd, all explicit modes obtain additional nodes due to their deformation with $\bar{\psi}$ increase, while if n is even, some modes deform with no change in the number of nodes; (ii) as a rule, the normal modes acquire their maximum number of nodes, $N^{(i)}(\infty)$, when $\bar{\psi} \leqslant 10^4$; (iii) several non-consecutive modes may have the same number of nodes; and (iv) the sequence of $N^{(i)}(\bar{\psi})$, $i = 1, \ldots, n$ is not ordered in subdomain 3.

3.3. To analyze the conventional eigenpairs (μ_k, Φ_k), $k = 1, \ldots, n$, we return to Figs 4 and 5. Clearly, the fundamental eigenfrequency, $\mu_1(\bar{\psi})$, is presented in Fig. 5b by the lowest enveloped curve $A_{12}A_{23}A_{34}$, the first overtone, $\mu_2(\bar{\psi})$, by the second enveloped curve $A_{12}A_{13}A_{23}A_{24}A_{34}$ and so on. Therefore, if $\bar{\psi} \in [0, \bar{\psi}_1^{cr})$,

$$\mu_1(\bar{\psi}) = \mu^{(1)}(\bar{\psi}) < \mu_2(\bar{\psi}) = \mu^{(2)}(\bar{\psi}) < \cdots < \mu_n = \mu^{(n)} = \pi \tag{54}$$

and

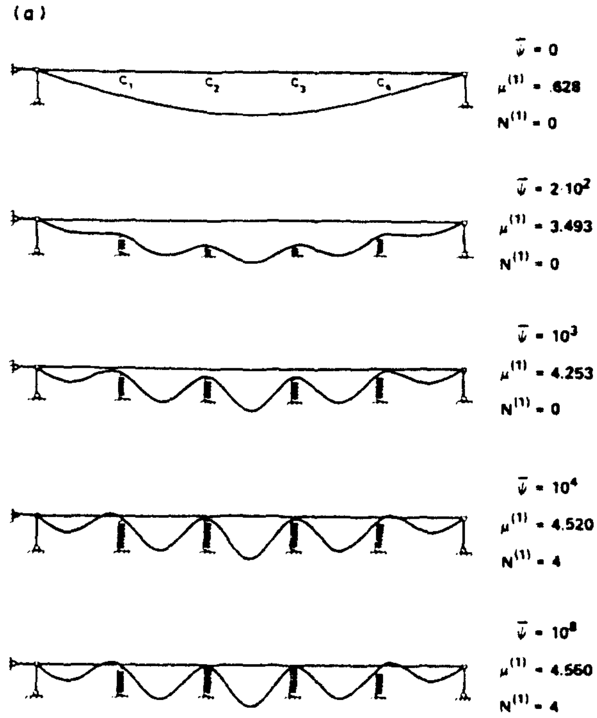


Fig. 6a. Mode $\Phi^{(1)}((\bar{\psi})$, $n = 5$.

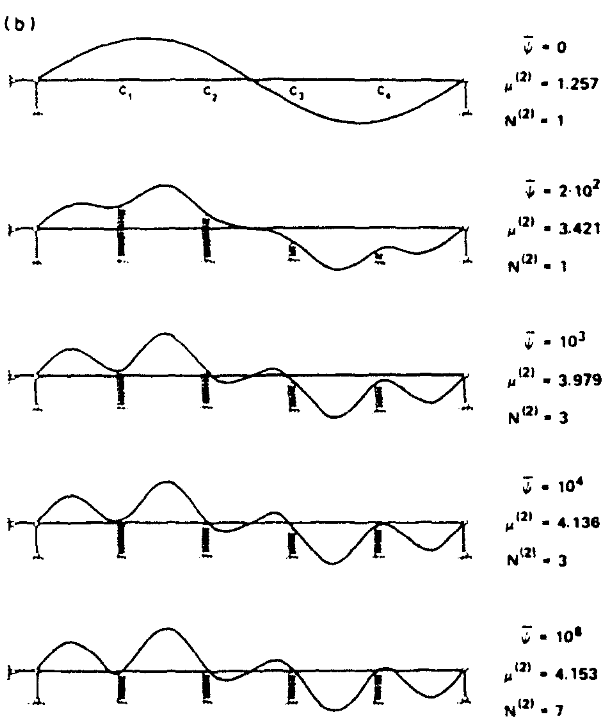


Fig. 6b. Mode $\Phi^{(2)}(\bar{\psi})$, $n = 5$.

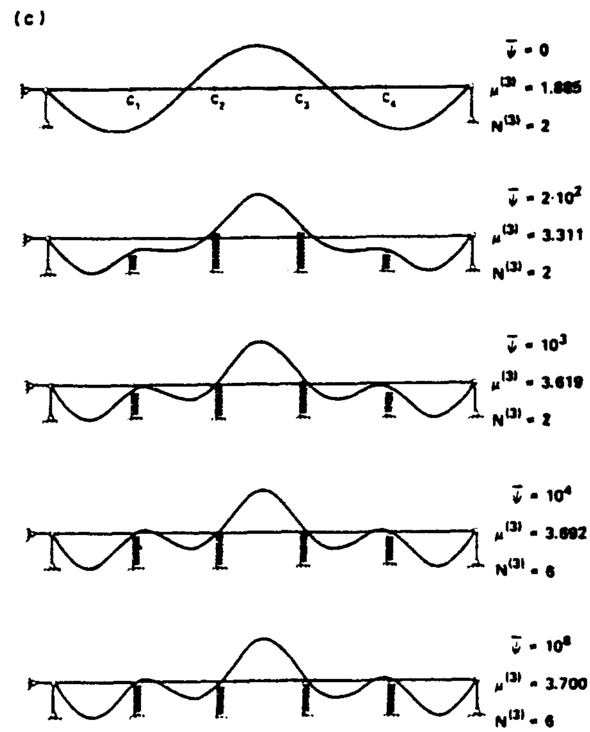


Fig. 6c. Mode $\Phi^{(3)}(\bar{\psi})$, $n = 5$.

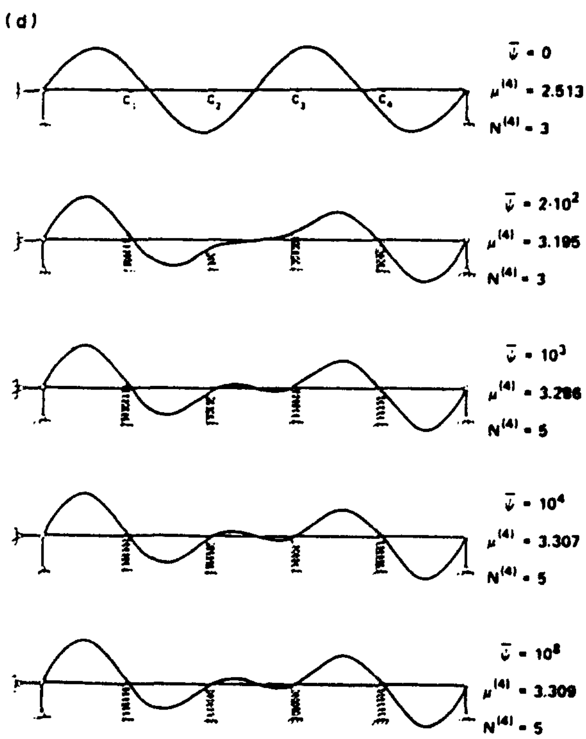


Fig. 6d. Mode $\Phi^{(4)}(\bar{\psi})$, $n = 5$.

Table 4. Number of nodes $N^{(i)}(\bar{\psi})$ of physical modes $\Phi^{(i)}(\bar{\psi})$

NUMBER OF SPANS	MODE $\Phi^{(i)}(\bar{\psi})$	$N^{(i)}(\bar{\psi})$				
		$\bar{\psi}=0$	$\bar{\psi}=10^3$	$\bar{\psi}=10^4$	$\bar{\psi}=10^5$	$\bar{\psi}=10^7$
11	1	0	0	4	8	8
	2	1	3	7	11	11
	3	2	6	10	10	10
	4	3	5	13	13	13
	5	4	8	12	12	12
	6	5	7	11	11	11
	7	6	10	10	10	10
	8	7	9	13	13	13
	9	8	8	12	12	12
	10	9	11	11	11	11
	11	10	10	10	10	10
12	1	0	0	4	8	8
	2	1	1	9	9	9
	3	2	2	14	14	14
	4	3	3	19	19	19
	5	4	8	12	12	12
	6	5	5	5	5	5
	7	6	10	14	14	14
	8	7	15	15	15	15
	9	8	8	8	8	8
	10	9	9	9	9	9
	11	10	10	10	10	10
	12	11	11	11	11	11

$$\Phi_k(\bar{\psi}) = \Phi^{(k)}(\bar{\psi}), \quad N_k(\bar{\psi}) = N^{(k)}(0) = k-1, \quad k = 1,\ldots,n. \tag{55}$$

Thus, in subdomain 1 the RBES possess the oscillatory properties.

If $\bar{\psi} \in [\bar{\psi}_1^{cr}, \bar{\psi}_2^{cr}]$, then

$$\left.\begin{aligned} \mu_1(\bar{\psi}) &= \mu^{(i)}(\bar{\psi}) \\ \Phi_1(\bar{\psi}) &= \Phi^{(i)}(\bar{\psi}) \\ N_1(\bar{\psi}) &= N^{(i)}(0) = i-1 \end{aligned}\right\} \quad \text{for } \bar{\psi}_{i-1,i} < \bar{\psi} < \bar{\psi}_{i,i+1}, \quad i = 2,\ldots,n-1. \tag{56}$$

Hence, as $\bar{\psi}$ increases, the fundamental mode, $\Phi_1(\bar{\psi})$, consequently coincides with $\Phi^{(2)}(\bar{\psi})$, $\Phi^{(3)}(\bar{\psi}),\ldots,\Phi^{(n-1)}(\bar{\psi})$, and therefore

$$1 \leqslant N_1(\bar{\psi}) \leqslant n-2. \tag{57}$$

When $\bar{\psi}$ passes through corresponding double frequency points $\bar{\psi}_{12}, \bar{\psi}_{23},\ldots,\bar{\psi}_{n-1,n}$, $N_1(\bar{\psi})$ instantaneously increases by one.

The eigenfrequency $\mu_k(\bar{\psi})$ is presented in Fig. 5b by the kth (from the bottom to the top) curve $A_{1k}A_{1,k+1}A_{2,k+1}\ldots A_{n-k,n}$. Hence the normal mode $\Phi_k(\bar{\psi})$ consequently coincides with $\Phi^{(k)}(\bar{\psi})$, $\Phi^{(1)}(\bar{\psi})$, $\Phi^{(k+1)}(\bar{\psi})$, $\Phi^{(2)}(\bar{\psi})$, $\Phi^{(k+2)}(\bar{\psi})$, etc. Corresponding double frequency points are determined by $i_1 - i = k-1$ or k, and

$$N_k(\bar{\psi}_{ii_1}+0) = N_k(\bar{\psi}_{ii_1}-0) + \begin{cases} 1-k, & \text{if } i_1 - i = k-1 \\ k, & \text{if } i_1 - i = k \end{cases}. \tag{58}$$

Thus, as $\bar{\psi}$ grows, $N_k(\bar{\psi})$ periodically decreases and increases, and some integers will occur twice, if $k \leqslant E((n+1)/2)$, where $E(x)$ is the greatest integer of x, Table 5. It becomes apparent that in subdomain 2 the fundamental mode has at least one node, while any higher mode may have even zero nodes. Hence in subdomain 2 the RBES have no oscillatory properties.

Table 5. Number of nodes $N_k(\bar{\psi})$ of conventional modes $\Phi_k(\bar{\psi})$, subdomains 1 and 2

NORMAL MODE $\Phi_k(\bar{\psi})$	$N_k(\bar{\psi}),\ \bar{\psi} \le \bar{\psi}_2^{cr} + 0$																			
1	0		1		2		3		4		5		6		7		8		9	
2	1	0	2	1	3	2	4	3	5	4	6	5	7	6	8	7	9	8		
3	2	0	3	1	4	2	5	3	6	4	7	5	8	6	9	7				
4	3	0	4	1	5	2	6	3	7	4	8	5	9	6						
5	4	0	5	1	6	2	7	3	8	4	9	5								
6	5	0	6	1	7	2	8	3	9	4										
7	6	0	7	1	8	2	9	3												
8	7	0	8	1	9	2														
9	8	0	9	1																
10	9	0																		

Whenever $\bar{\psi} > \bar{\psi}_2^{cr}$, all $\mu_k(\bar{\psi})$ are distinct again, but both sequences $\mu_k(\bar{\psi})$ and $\mu^{(k)}(\bar{\psi})$, $k = 1, \ldots, n$ are arranged in an opposite order, Fig. 4:

$$\mu_1 = \mu^{(n)} = \pi < \mu_2(\bar{\psi}) = \mu^{(n-1)}(\bar{\psi}) < \cdots < \mu_n(\bar{\psi}) = \mu^{(1)}(\bar{\psi}). \tag{59}$$

Therefore,

$$\Phi_k(\bar{\psi}) = \Phi^{(n+1-k)}(\bar{\psi}), \quad N_k(\bar{\psi}) = N^{(n+1-k)}(\bar{\psi}), \quad k = 1, \ldots, n. \tag{60}$$

In subdomain 3 the normal modes do not possess the oscillatory properties, since in this subdomain the sequence of $N^{(k)}(\bar{\psi})$ is not ordered. Thus we reach the following results:

Theorem 3. (i) The fundamental mode $\Phi_1(\bar{\psi})$, may have any number of nodes up to $n-1$.† When increasing $\bar{\psi}$ passes through the corresponding double frequency points $\bar{\psi}_{12}, \bar{\psi}_{23}, \ldots, \bar{\psi}_{n-1,n}$, its number of nodes increases by one, changing the mode's symmetry.

(ii) Each higher mode, $\Phi_k(\bar{\psi})$, $k = 2, \ldots, n$, may have any number of nodes from zero up to $n-1$.

(iii) Several non-consecutive normal modes may have the same number of nodes.

Theorem 4. For each fixed number of spans n, $\bar{\psi}_1^{cr}$ (49) and $\bar{\psi}_2^{cr}$ (50) divide the entire domain $[0, \infty)$ into three subdomains $[0, \bar{\psi}_1^{cr})$, $[\bar{\psi}_1^{cr}, \bar{\psi}_2^{cr}]$, and $(\bar{\psi}_2^{cr}, \infty)$ so that the RBES (i) possess the oscillatory properties in subdomain 1, (ii) have no such properties in subdomain 2, and (iii) have distinct eigenfrequencies in subdomain 3.

3.4. The eigenfrequency $\mu_k(\bar{\psi}, n)$ can be easily found graphically. Present $\mu^{(i)}(\bar{\psi}, n)$ in the form

$$\mu^{(i)}(\bar{\psi}, n) \equiv \mu_{\bar{\psi}}(\cos\theta_i), \quad \theta_i = i\pi/n, \quad i = 1, \ldots, n \tag{61}$$

They are shown in Fig. 7 for the first zone of frequency condensation, $i \leqslant n$. Suppose we are interested in a particular $\mu_k(\bar{\psi}, n)$ for given n and $\bar{\psi}$. Then, if $\bar{\psi} < \bar{\psi}_1^{cr}$ (Table 3), one should compute $\cos\theta_k$ and use Fig. 7. If $\bar{\psi} > \bar{\psi}_2^{cr}$, it is necessary to compute $\cos\theta_{n+1-k}$. When $\bar{\psi}$ belongs to subdomain 2, one should find all n values $\mu_{\bar{\psi}}(\cos\theta_i)$, $i = 1, \ldots, n$, and choose the kth.

† The analogous fact is well known in the theory of elastic stability (Timoshenko, 1936): a simply supported rectangular plate, uniformly compressed along the short sides, may have one, two, etc., half-waves in the critical state (the fundamental mode), depending on the ratio of its sides.

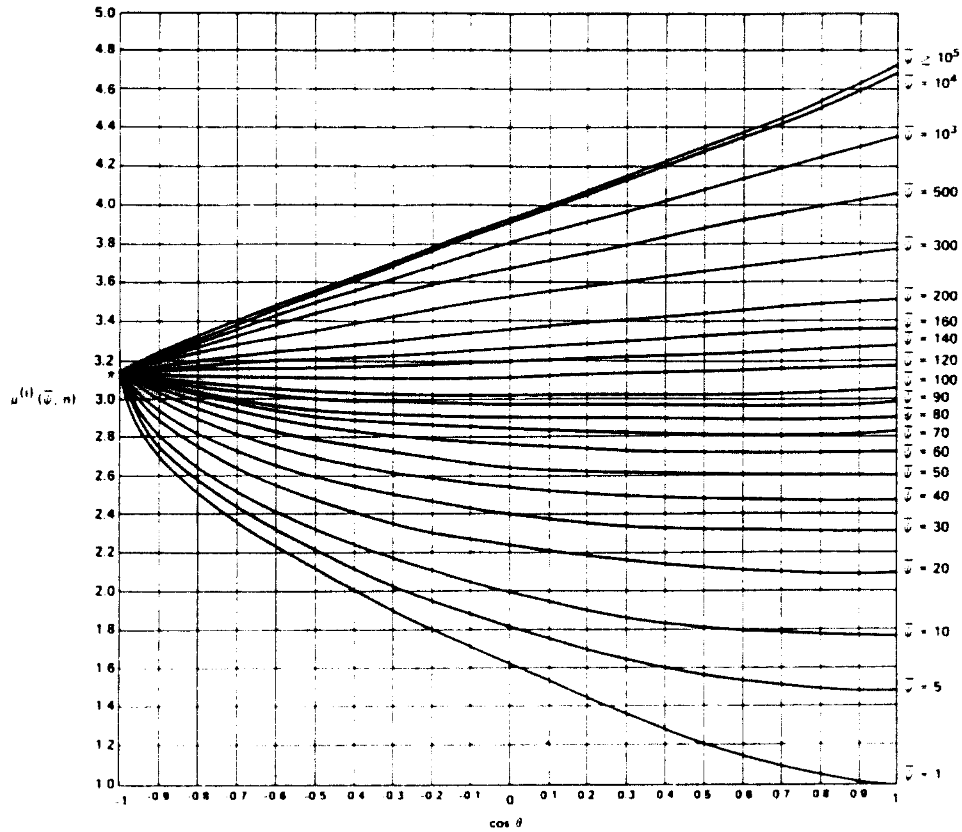


Fig. 7. Functions $\mu^{(i)}(\bar\psi, n)$, first zone of frequency condensation.

It follows from Fig. 7 that in subdomain 1 $\mu(\bar\psi)|_{\cos\theta=1} \leqslant \mu_k(\bar\psi, n) \leqslant \pi$, while in subdomain 3, $\pi \leqslant \mu_k(\bar\psi, n) \leqslant 4.73$, where $\mu = 4.73$ is the fundamental eigenfrequency of a one-span beam with fixed ends. In subdomain 2, $\mu_1(\bar\psi, n) \simeq \mu_1(\bar\psi)$. In general, with n increasing the RBES eigenfrequencies decrease.

4. MODAL RESPONSE FUNCTIONS OF THE RBES

4.1. The modal response functions, $\gamma_{p,k}$ (27), are associated with the conventional normal modes, $\Phi_k(\bar\psi)$. We call them the *conventional* modal response functions. As in Section 3, we begin with physical modes, $\Phi^{(i)}(\bar\psi)$, and introduce the corresponding *physical* modal response functions

$$\gamma_p^{(i)} = [\Gamma_p^{(i)}/\Gamma_p]^2, \quad p = 0, 1, 2, 4;\ i = 1, \ldots, n \tag{62}$$

where

$$\Gamma_p^{(i)} = \Gamma_0^{(i)}/(\mu^{(i)})^p, \quad p = 0, 1, 2, 4;\ i = 1, \ldots, n. \tag{63}$$

Due to a given excitation and n fixed, functions $\gamma_p^{(i)}(\bar\psi)$ are continuous in the entire domain, Fig. 8. Hence the RBES total response functions, $\Gamma_p(\bar\psi)$, $p = 1, 2, 4$, eqns (22) and (26), are also continuous. They monotonically decrease to their asymptotic values, $\Gamma_p(\infty)$.

Observation of Fig. 8 shows that (i) the unweighted participation factor $\Gamma_0^{(i)}$, cannot describe three different modal contributions (in w_{max}, in M_{max} and in V_{max}), moreover, it does not estimate either one, and (ii) a common statement that "the contribution of the same mode increases from displacement to moment to shear" is incorrect.

Inspection of Fig. 8 also reveals the existence of an additional critical point, $\bar\psi_3^{cr}$, such that

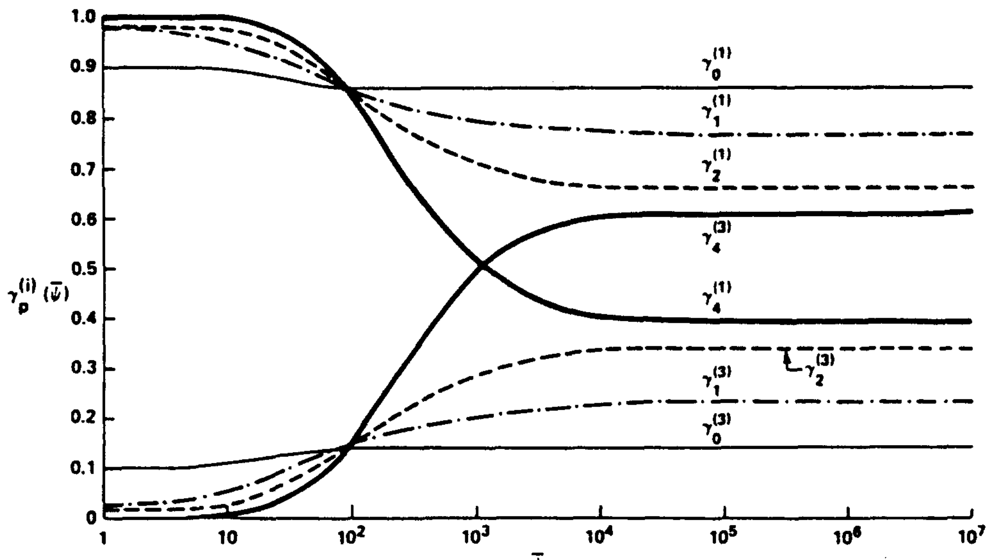


Fig. 8. Functions $\gamma_p^{(i)}(\bar{\psi})$, $n = 4$.

$$|\gamma_p^{(i)}(\bar{\psi}_3^{cr}) - \gamma_p^{(i)}(\infty)| < \varepsilon, \quad \varepsilon > 0\text{—small}, \quad i = 1, \ldots, n. \tag{64}$$

Since $\bar{\psi}_3^{cr}$ is the same for all modes,

$$|\Gamma_p(\bar{\psi}_3^{cr}) - \Gamma_p(\infty)| < \varepsilon_1, \quad \varepsilon_1 > 0\text{—small}, \quad p = 0, 1, 2, 4. \tag{65}$$

Regardless of n, it was found that $\Gamma_4(10^4)/\Gamma_4(\infty) \leqslant 1.028$, $\Gamma_2(10^4)/\Gamma_2(\infty) \leqslant 1.021$, and $\Gamma_1(10^4)/\Gamma_1(\infty) \leqslant 1.015$. Hence $\bar{\psi}_3^{cr} \simeq 10^4$. Note also that $\max_k(\mu_k(10^4)/\mu_k(\infty)) \geqslant 0.991$ and $\max_{i,j} w_j^{(i)}(10^4) \leqslant 0.025$. Usually, but not always, $N^{(i)}(10^4) = N^{(i)}(\infty)$, Table 4. With respect to n, Γ_ps increase in subdomain 1 (if $\bar{\psi} = 0$, $\Gamma_p = (4/\pi^{p+1})n^p$) and decrease in subdomain 3. Hence in subdomain 1 the RBES behave as one-span beams, while if $\bar{\psi} \geqslant \bar{\psi}_3^{cr}$ they are similar to the RBRS.

4.2. In contrast with $\gamma_p^{(i)}(\bar{\psi})$, functions $\gamma_{p,k}(\bar{\psi})$ are continuous only in subdomains 1 and 3. They have discontinuities at double frequency points. To prove this, let us consider two physical eigenpairs $(\mu^{(i)}, \Phi^{(i)})$ and $(\mu^{(i_1)}, \Phi^{(i_1)})$ in the neighborhood of point $\bar{\psi}_{ii_1}$, Fig. 9. Suppose that at $\bar{\psi} = \bar{\psi}_{ii_1} - 0$, $\mu^{(i)}$ and $\mu^{(i_1)}$ are the kth and the k_1th conventional eigenfrequencies, respectively, and $k_1 > k$:

$$\mu^{(i)}(\bar{\psi}_{ii_1} - 0) = \mu_k < \mu_{k_1} = \mu^{(i_1)}(\bar{\psi}_{ii_1} - 0).$$

Then, when $\bar{\psi}$ passes through $\bar{\psi}_{ii_1}$, they exchange their locations in the spectrum

$$\mu^{(i)}(\bar{\psi}_{ii_1} + 0) = \mu_{k_1} > \mu_k = \mu^{(i_1)}(\bar{\psi}_{ii_1} + 0).$$

Hence

$$\Phi_k(\bar{\psi}_{ii_1} - 0) = \Phi^{(i)}(\bar{\psi}_{ii_1} - 0), \quad \Phi_{k_1}(\bar{\psi}_{ii_1} - 0) = \Phi^{(i_1)}(\bar{\psi}_{ii_1} - 0)$$

and

$$\Phi_k(\bar{\psi}_{ii_1} + 0) = \Phi^{(i_1)}(\bar{\psi}_{ii_1} + 0), \quad \Phi_{k_1}(\bar{\psi}_{ii_1} + 0) = \Phi^{(i)}(\bar{\psi}_{ii_1} + 0).$$

Thus in the neighborhood of $\bar{\psi}_{ii_1}$ the conventional modal response functions $\gamma_{p,k}(\bar{\psi})$ and $\gamma_{p,k_1}(\bar{\psi})$ are

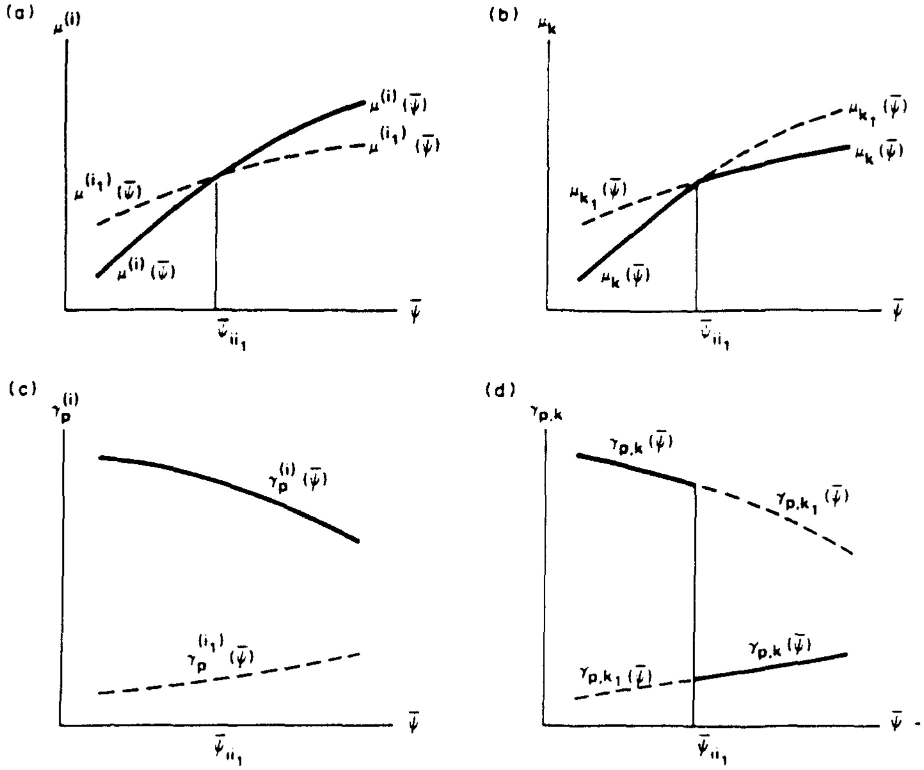


Fig. 9. Discontinuity of modal response functions $\gamma_{p,k}(\bar{\psi})$ and $\gamma_{p,k_1}(\bar{\psi})$.

$$\gamma_{p,k}(\bar{\psi}) = \begin{cases} \gamma_p^{(i)}(\bar{\psi}) \\ \gamma_p^{(i_1)}(\bar{\psi}) \end{cases}, \quad \gamma_{p,k_1}(\bar{\psi}) = \begin{cases} \gamma_p^{(i_1)}(\bar{\psi}), & \text{if } \bar{\psi} < \bar{\psi}_{ii_1} \\ \gamma_p^{(i)}(\bar{\psi}), & \text{if } \bar{\psi} > \bar{\psi}_{ii_1} \end{cases}, \quad k_1 > k. \tag{66}$$

Since $\gamma_p^{(i)}(\bar{\psi})$ and $\gamma_p^{(i_1)}(\bar{\psi})$ are not equal at any point $\bar{\psi} \in [\bar{\psi}_1^{cr}, \bar{\psi}_2^{cr}]$, Fig. 8, the double frequency point $\bar{\psi}_{ii_1}$ is a discontinuity point of $\gamma_{p,k}(\bar{\psi})$ and $\gamma_{p,k_1}(\bar{\psi})$. Hence small changes in $\bar{\psi}$ may lead to significant (even total, if $\gamma_p^{(i)}$ or $\gamma_p^{(i_1)}$ equal to zero) changes in the modal responses due to the same excitation, Fig. 10.

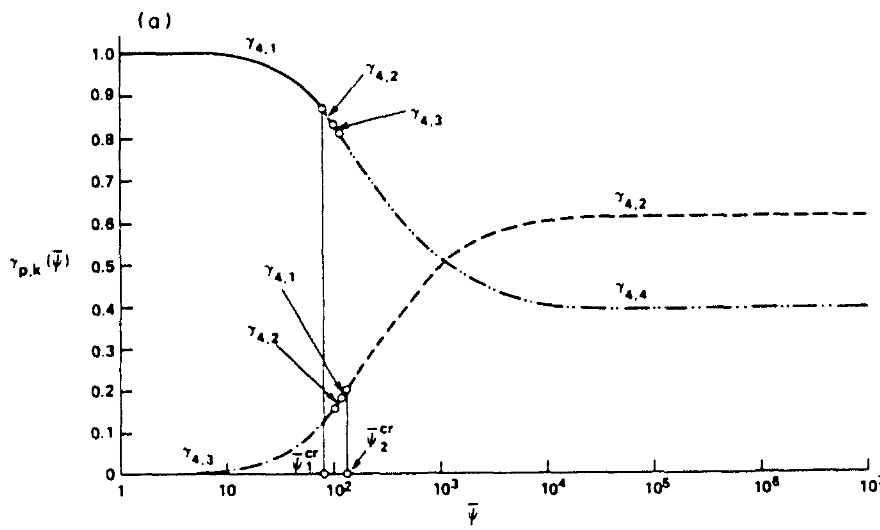


Fig. 10a. Functions $\gamma_{4,k}(\bar{\psi})$, $n = 4$.

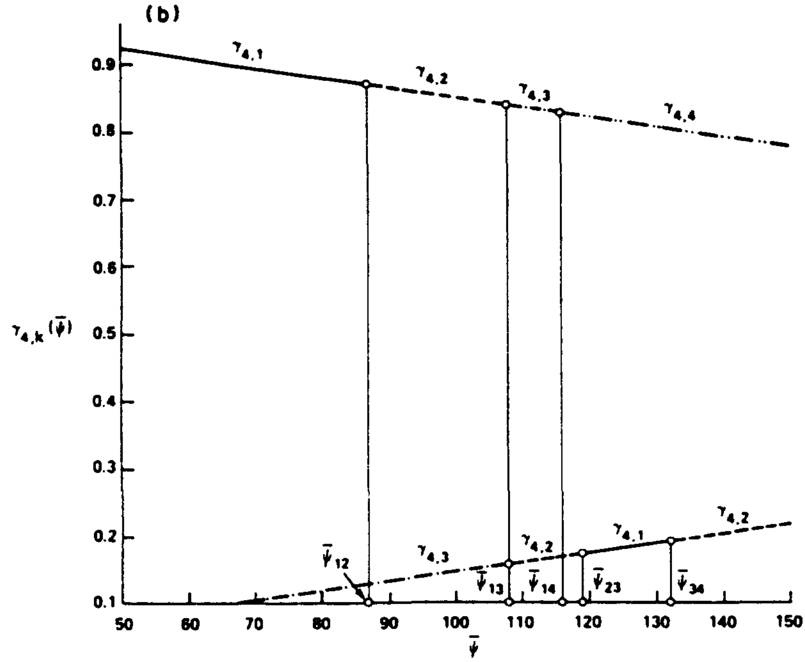


Fig. 10b. Functions $\gamma_{4,k}(\bar{\psi})$, $n = 4$, subdomains 1 and 2.

4.3. Present the above results as following theorems:

Theorem 5. The physical modal response functions, $\gamma_p^{(i)}(\bar{\psi})$, $p = 1, 2, 4$, $i = 1, \ldots, n$, are continuous in the entire domain $[0, \infty)$. If for a given $\bar{\psi}$ the physical mode $\Phi^{(i)}(\bar{\psi})$, $i = 1, \ldots, n$, is excited, it will be excited in the entire domain due to the same excitation.

Theorem 6. Each conventional modal response function, $\gamma_{p,k}(\bar{\psi})$, $p = 1, 2, 4$, $k = 1, \ldots, n$, has $n-1$ discontinuity points. They belong to subdomain 2 and coincide with double frequency points $\bar{\psi}_{ii_1}$, $i_1 > i = 1, \ldots, n-1$. Small changes in $\bar{\psi}$ may lead to significant (even complete) changes in the responses of the conventional modes $\Phi_k(\bar{\psi})$ due to the same excitation.

5. DISCUSSION

5.1. The class of linear mechanical structures $S(\bar{\psi})$ that have no oscillatory properties is not limited by the RBES, which were chosen as the object of the above analysis only to simplify the effort of constructing an unconventional (physical) sequence of eigenpairs. The physical eigenfrequencies $\mu^{(1)}(\bar{\psi})$ and $\mu^{(2)}(\bar{\psi})$ corresponding to irregular symmetric three- and four-span beams with elastic interior supports are shown in Fig. 11. One can see that in the first case (three-span beam, Fig. 11a) both curves intersect twice, in the second (four-span beam, Fig. 11b)—three times. Hence as $\bar{\psi}$ grows the conventional modes $\Phi_1(\bar{\psi})$ and $\Phi_2(\bar{\psi})$ change their symmetry several times and the corresponding modal response functions, $\gamma_{p,1}(\bar{\psi})$ and $\gamma_{p,2}(\bar{\psi})$, $p = 1, 2, 3, 4$, have several discontinuity points.

5.2. In accordance with variational principles, some eigenfrequencies of an arbitrary linear mechanical structure $S(\bar{\psi})$ increase as a rigidity parameter $\bar{\psi}$ increases (explicit $\mu^{(i)}(\bar{\psi})$), while others remain unchanged (implicit eigenfrequencies). This is true regardless of the order of ODEs, which describe free vibration of structures $S(\bar{\psi})$. With respect to the RBES the rate of $\mu^{(i)}(\bar{\psi})$ growth can be found from (38):

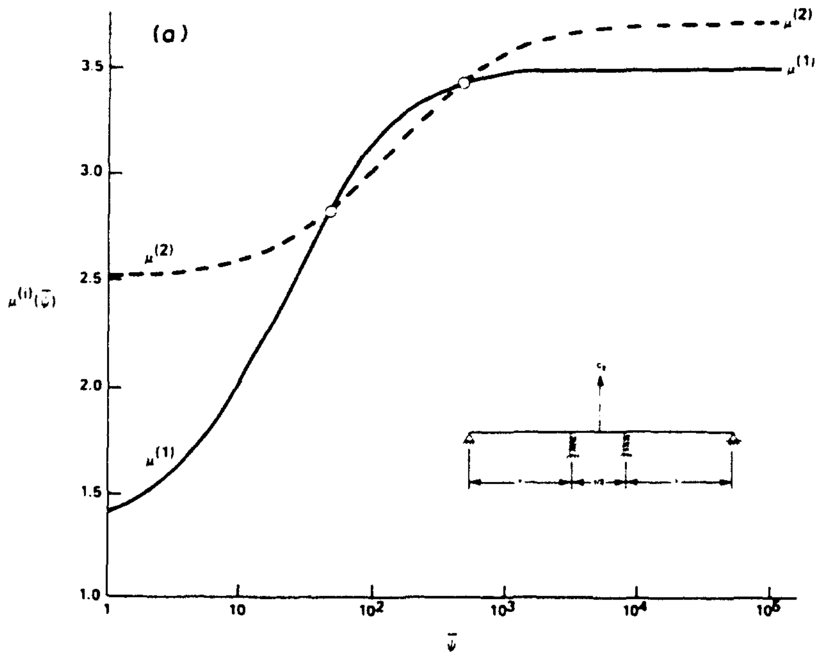


Fig. 11a. Functions $\mu^{(1)}(\bar{\psi})$ and $\mu^{(2)}(\bar{\psi})$ for an irregular symmetric three-span beam.

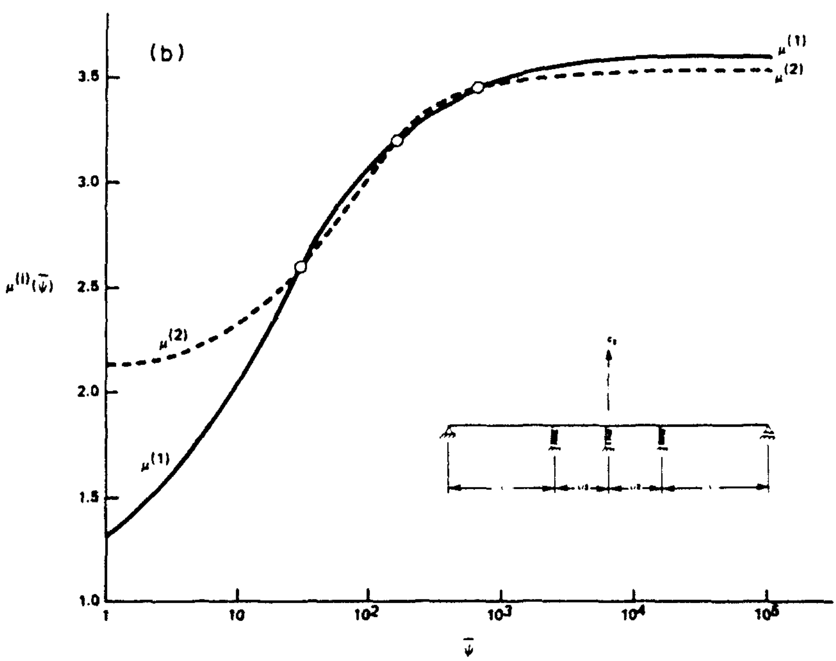


Fig. 11b. Functions $\mu^{(1)}(\bar{\psi})$ and $\mu^{(2)}(\bar{\psi})$ for an irregular symmetric four-span beam.

$$\frac{d\mu^{(i)}}{d\bar{\psi}} = 4\mu^{(i)^3}\Big/\bar{\psi}^2\left[12\frac{\mu^{(i)^2}}{\bar{\psi}} + \frac{1-\cos\mu^{(i)}\cos\theta_i}{(\cos\mu^{(i)}-\cos\theta_i)^2} + \frac{\cosh\mu^{(i)}\cos\theta_i - 1}{(\cosh\mu^{(i)}-\cos\theta_i)^2}\right],$$
$$\theta_i = i\pi/n, \quad i = 1,\ldots,n-1. \quad (67)$$

Hence for any $\bar{\psi} < \infty$

$$\frac{d\mu^{(1)}}{d\bar{\psi}} > \frac{d\mu^{(2)}}{d\bar{\psi}} > \cdots > \frac{d\mu^{(n)}}{d\bar{\psi}} = 0. \tag{68}$$

As $\bar{\psi}$ increases all $d\mu^{(i)}/d\bar{\psi}$ decrease monotonically and almost equidistantly. Thus, each pair of the RBES eigenfrequencies intersects only once. In general, physical eigenfrequencies of $S_4(\bar{\psi})$, that is, of $S(\bar{\psi})$ governed by fourth order ODEs, may intersect each other several times, and (68) is a sufficient condition for the existence of multiple eigenfrequencies. As an example of $S_2(\bar{\psi})$ structures, consider longitudinal vibration of regular beams with elastic springs between spans. Corresponding decoupled frequency equations are:

$$\mu(\cos\mu+\cos\theta_i)+\tfrac{1}{2}\bar{\psi}\sin\mu = 0,\quad \theta_i = i\pi/n,\quad i = 1,\dots,n-1 \tag{69}$$

where $\mu = (m\omega^2 l^2/EA)^{1/2}$, A is the cross-sectional area, and other quantities are the same as in (4). Thus

$$\frac{d\mu^{(i)}}{d\bar{\psi}} = -\frac{1}{2}\frac{\sin\mu^{(i)}}{(1+\frac{1}{2}\bar{\psi})\cos\mu^{(i)}-\mu^{(i)}\sin\mu^{(i)}+\cos\theta_i},\quad i = 1,\dots,n-1. \tag{70}$$

and for any finite $\bar{\psi}$

$$\frac{d\mu^{(1)}}{d\bar{\psi}} < \frac{d\mu^{(2)}}{d\bar{\psi}} < \cdots < \frac{d\mu^{(n-1)}}{d\bar{\psi}}. \tag{71}$$

This is true for any $S_2(\bar{\psi})$, in fact, eqn (71) is the necessary and sufficient condition for the existence of distinct eigenfrequencies.

Inequalities (71) and (68) express the principal difference between spectral properties of $S_2(\bar{\psi})$ and $S_4(\bar{\psi})$. It follows from (71) that structures $S_2(\bar{\psi})$ always possess the oscillatory properties, while (68) leads to the conclusion that the existence of the oscillatory properties of structures $S_4(\bar{\psi})$ should be treated as the exception rather than the rule. Thus a common association of higher modes of mechanical structures with higher number of nodes, which is true for $S_2(\bar{\psi})$, is incorrect in general.

5.3. Apparently, there is no better rule for eigenpair labeling than the conventional one, but it leads to discontinuities of the modal response functions $\gamma_{p,k}(\bar{\psi})$ and directly affects the problem of modal truncation, a concept of great value in the modal analysis. In some cases the modal truncation is based on a specified magnitude of the natural frequency: all normal modes whose frequencies do not exceed this value are preserved. Discontinuities of $\gamma_{p,k}(\bar{\psi})$ do not affect the results in these cases. However, quite often we have no such criterion. Therefore we usually limit the modal analysis to the first m normal modes choosing m from the previous experience. In accordance with Theorem 6 this may lead to serious (sometimes critical) errors and special precautions are required.

If the excitation of $S_4(\bar{\psi})$ is given by $p(x,t) = q(x)f(t)$, eqn (9), the following procedure may be recommended:

(i) apply $q(x)$ statically and compute displacements $w(x)$ and moments $M(x)$,
(ii) find the Rayleigh quotient

$$\omega_R^2 = \int_0^L q(x)w(x)\,dx \Big/ \int_0^L m(x)w^2(x)\,dx \tag{72}$$

or the Grammel quotient which is less conservative

$$\omega_G^2 = \int_0^L m(x)w^2(x)\,\mathrm{d}x \Big/ \int_0^L \frac{M^2(x)}{EI(x)}\,\mathrm{d}x. \tag{73}$$

(iii) include in the modal superposition all normal modes whose frequencies do not exceed ω_R or $\omega_G < \omega_R$.

Acknowledgements—The author wishes to thank Prof. Peter Lax of the Courant Institute of Mathematical Sciences, Prof. Edward N. Kuznetsov of the University of Illinois at Urbana-Champaign and Dr Gregory Raynus of EBASCO Services Inc., for their valuable comments on, and discussion of the paper.

REFERENCES

Biggs, J. M. (1964). *Introduction to Structural Dynamics*. McGraw-Hill, New York.

Courant, R. and Hilbert, D. (1953). *Methods of Mathematical Physics*. Vol. I. John Wiley & Sons, New York.

Dinkevich, S. (1974). Explicit spectral decomposition of quasi regular matrices and analysis of regular structures. In *Research in the Theory of Structures*, Vol. 20, pp. 216–233. Stroiizdat, Moscow.

Dinkevich, S. (1977). *Analysis of Cyclicly Symmetric Structures: The Spectral Method*. Stroiizdat, Moscow.

Dinkevich, S. (1986). Explicit block diagonal decomposition of block matrices corresponding to symmetric and regular structures of finite size. Courant Institute of Mathematical Sciences, Report MF-108, New York University.

Gantmacher, F. R. and Krein, M. G. (1950). *Oscillation Matrices and Kernels and Small Vibrations of Dynamic Systems*, 2nd Edn. Gostekhizdat, Moscow.

Gantmacher, F. R. (1960). *The Theory of Matrices*, Vols I and 2. Chelsea Publishing Co., New York.

Leites, S. D. (1974). Spectral function in the problem of free vibration of beam systems. In *Research in the Theory of Structures*, Vol. 20, pp. 91–101. Stroiizdat, Moscow.

Meirovitch, L. (1967). *Analytical Methods in Vibration*. The MacMillan Co., London.

Timoshenko, S. (1936). *Theory of Elastic Stability*. McGraw-Hill, New York.